

LECTURE 1: METRIC SPACES, TREES, AND FREE GROUPS

CHRISTOPHER H. CASHEN

1. MOTIVATION AND METRIC SPACES

Goal. Study groups G (usually finitely presented) via the geometry of their actions. Finitely presented groups are countable and discrete, so tools from the theory of topological groups are not directly useful. Instead we focus on making the group act on a metric space.

- If a group has a sufficiently nice action on a metric space, the geometry of that space influences the group's properties.
- Negative curvature (hyperbolicity) is a group property.
- Hyperbolic groups have a well-defined boundary at infinity.

$$\begin{array}{ccc} \text{dynamics on the boundary} & \longleftrightarrow & \text{geometry of the space} \\ & & \longleftrightarrow \text{algebraic structure of the group} \end{array}$$

Definition 1.1. A *metric space* (X, d) consists of a set X and a function $d: X \times X \rightarrow \mathbb{R}$ satisfying, for all $x, y, z \in X$:

Positive definiteness: $d(x, y) \geq 0$, with equality if and only if $x = y$.

Symmetry: $d(x, y) = d(y, x)$

Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

The metric topology has as a basis the open balls:

$$B_r(x) := \{y \in X \mid d(x, y) < r\}$$

Also define the closed balls and open and closed neighborhoods of subsets:

$$\bar{B}_r(x) := \{y \in X \mid d(x, y) \leq r\} \quad N_r(Y) := \{x \in X \mid d(x, Y) < r\} \quad \bar{N}_r(Y) := \{x \in X \mid d(x, Y) \leq r\}$$

In the latter two we mean $d(x, Y) := \inf_{y \in Y} d(x, y)$.

2. PATHS, LENGTH SPACES, AND GEODESICS

A *path* is a continuous map $\gamma: I \rightarrow X$, where $I \subseteq \mathbb{R}$ is an interval.

The *length* of a path $\gamma: [a, b] \rightarrow X$ is:

$$|\gamma| := \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \mid a = t_0 < t_1 < \cdots < t_n = b \right\}$$

A path is *rectifiable* if its length is finite, and *locally rectifiable* if its restriction to every compact subinterval of its domain has finite length.

Lemma 2.1 (Properties of length). *For a locally rectifiable path $\gamma: I \rightarrow X$ and $r \leq s \leq t$ in I :*

- $d(\gamma(s), \gamma(t)) \leq |\gamma|_{[s, t]}$
- $|\gamma|_{[r, s]} + |\gamma|_{[s, t]} = |\gamma|_{[r, t]}$

- *There is a unit-speed reparametrization:* For each $[s, t] \subseteq I$ there is a continuous, weakly monotonic function $\phi: [s, t] \rightarrow [0, |\gamma|_{[s, t]}]$ and a unique path $\tilde{\gamma}: [0, |\gamma|_{[s, t]}] \rightarrow X$ such that $\tilde{\gamma} \circ \phi = \gamma$ and $|\tilde{\gamma}|_{[0, u]} = u$ for all $u \in [0, |\gamma|_{[s, t]}]$.

Definition 2.2. A metric space (X, d) is a *length space* if, for all $x, y \in X$ and every $\varepsilon > 0$, there exists a rectifiable path γ from x to y of length less than $d(x, y) + \varepsilon$.

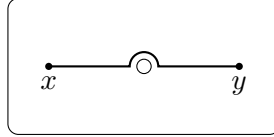
A *geodesic* is a locally rectifiable path $\gamma: I \rightarrow X$ such that $\forall s, t \in I$:

$$|\gamma|_{[s, t]} = d(\gamma(s), \gamma(t))$$

Definition 2.3. A metric space (X, d) is a *geodesic space* if every pair $x, y \in X$ can be joined by a geodesic.

Example 2.4. A space that is not path-connected is not a length space.

Example 2.5 (Punctured plane). Consider the plane minus a point, $\mathbb{R}^2 \setminus \{\vec{0}\}$, with the restriction of the Euclidean metric. For points x, y whose Euclidean segment passes through the puncture, any path between them in the punctured plane must detour around the puncture, which costs some additional length. We can make the detour as small as desired, but not zero. This space is a length space but not a geodesic space.



3. GRAPHS AND TREES

A *graph* Γ consists of a vertex set $V\Gamma$ and an edge set $E\Gamma$. Multiple edges and loops are allowed. Regard every edge as having length 1; the induced path-length distance is a metric precisely when Γ is connected, and then it is geodesic.

An *edge path* has the form:

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n$$

A *cycle* is a closed edge path ($v_0 = v_n$) without repeated edges or other repeated vertices.

Definition 3.1. A *tree* is a nonempty connected graph without cycles.

Lemma 3.2. A finite tree that is not a single vertex has at least two leaves, that is, vertices of valence 1.

Proof. Choose a longest embedded edge path $v_0 \xrightarrow{e_1} \cdots \xrightarrow{e_n} v_n$. We claim that v_0 and v_n are leaves. If, say, v_0 had another incident edge f with opposite endpoint w , then either $w \notin \{v_0, \dots, v_n\}$, giving a longer embedded path, or $w \in \{v_0, \dots, v_n\}$, giving a cycle. $\square$

Corollary 3.3. If Γ is a tree, then every nonconstant closed edge path backtracks at least twice (counted cyclically, so allowing the possibility that one of the backtracks is at the initial/terminal vertex).

Proof. The image is a connected finite subtree, hence has at least two leaves. At each leaf the path must backtrack. $\square$

Corollary 3.4. If Γ is a tree and $x \neq y$ are vertices, there is a unique embedded edge path from x to y .

Proof. Suppose α, β are embedded paths from x to y . Let $\bar{\beta}$ be the edge path β traversed in the opposite direction. The concatenation $\alpha + \bar{\beta}$ is a closed edge path, so it has at least two backtracks. Since α and β have no internal backtracking, these must occur at the endpoints x, y . Removing the common initial and terminal edges and inducting on $d(x, y)$ gives $\alpha = \beta$. $\square$

Lemma 3.5. *For a connected graph Γ , the following are equivalent:*

- (i) Γ is a tree.
- (ii) Γ has no embedded circle $\mathbb{S}^1$.
- (iii) Γ has no loops and every pair of vertices is joined by a unique embedded edge path.
- (iv) Every pair of points is joined by a unique geodesic.
- (v) For each point x , either $\Gamma = \{x\}$, or x is a leaf, or x is a cut point of Γ .

For points x, y of a tree, write $[x, y]$ for the unique geodesic joining them.

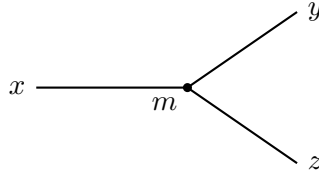
Lemma 3.6. *If Γ is a tree, $x, y \in \Gamma$, and γ is any path from x to y , then the image of γ contains $[x, y]$.*

Proof. The image of γ contains x and y . If $x = y$ we are done. Otherwise, every interior point z of $[x, y]$ is a cut point of Γ . If z is an interior point of an edge then locally it only has two complementary components, so these identify the two global complementary components. Thus, x and y are separated by z , meaning that every path from x to y passes through z . Since the image of γ is closed and contains the interior points of every edge on $[x, y]$, it also contains the vertices. $\square$

Lemma 3.7 (Tripod lemma). *For any x, y, z in a tree there exists a unique point m such that:*

$$[x, y] \cap [y, z] \cap [x, z] = \{m\}$$

Proof. Let m be the last common point of $[x, y]$ and $[x, z]$. The two geodesic segments $[y, m]$ and $[m, z]$ concatenate without backtracking, since a backtrack would contradict the choice of m . Thus their concatenation is $[y, z]$, and the triple intersection is exactly $\{m\}$.



$\square$

Theorem 3.8. *Geodesic triangles in a tree are 0-thin: if $x_0, x_1, x_2 \in \Gamma$ and γ_i is a geodesic from x_i to x_{i+1} (indices modulo 3), then for all i :*

$$\gamma_i \subseteq \gamma_{i+1} \cup \gamma_{i+2} = \bar{N}_0(\gamma_{i+1} \cup \gamma_{i+2})$$

Proof. Every triangle is a tripod by the preceding lemma. If m is its central point, then for each i :

$$[x_i, x_{i+1}] = [x_i, m] \cup [m, x_{i+1}] \subseteq [x_i, x_{i+2}] \cup [x_{i+2}, x_{i+1}] \quad \square$$

4. THE GROMOV PRODUCT AND THE BOUNDARY OF A TREE

Definition 4.1. The *Gromov product* of x and y based at z is:

$$(x \mid y)_z := \frac{1}{2} (d(x, z) + d(y, z) - d(x, y))$$

If T is a tree and m is the center of the tripod with vertices x, y, z , then

$$(x \mid y)_z = d(m, z) = d(z, [x, y])$$

Let T be a possibly unbounded tree. A (geodesic) *ray* is an embedded edge path that is infinite in one direction:

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{\quad} \cdots$$

Definition 4.2. Two rays are *asymptotic* if their intersection contains a ray, that is, if they eventually agree. The *boundary* ∂T is the set of asymptotic equivalence classes of rays.

Fix a base vertex $o \in T$.

Lemma 4.3. *For each $\xi \in \partial T$, there is exactly one ray starting at o and representing ξ .*

Proof. Take a ray $\alpha \in \xi$ and let $\alpha(t_0)$ be the last vertex of α on the segment $[o, \alpha(0)]$. Then $[o, \alpha(t_0)] + \alpha|_{[t_0, \infty)}$ is a ray representing ξ . Uniqueness follows from uniqueness of embedded paths in a tree. $\square$

Extending the Gromov product to ∂T . For $\xi, \eta \in \partial T$, define:

$$(\xi | \eta)_o := \text{length of the common initial segment of the rays } \alpha_\xi \in \xi \text{ and } \alpha_\eta \in \eta \text{ based at } o$$

Lemma 4.4. *If $\xi \neq \eta \in \partial T$, there is a unique bi-infinite geodesic (ξ, η) such that each of its two rays from any vertex represents one of ξ or η . Moreover,*

$$(\xi | \eta)_o = d(o, (\xi, \eta))$$

Theorem 4.5. *The following defines a metric on ∂T :*

$$d(\xi, \eta) := 2^{-(\xi | \eta)_o}$$

Proof. Symmetry and nonnegativity are immediate. Also,

$$d(\xi, \eta) = 0 \iff (\xi | \eta)_o = \infty \iff \alpha_\xi = \alpha_\eta \iff \xi = \eta$$

For the triangle inequality, the common initial segment of α_ξ and α_ζ has length at least the minimum of the common initial segment lengths for $(\alpha_\xi, \alpha_\eta)$ and $(\alpha_\eta, \alpha_\zeta)$. Thus:

$$(\xi | \zeta)_o \geq \min\{(\xi | \eta)_o, (\eta | \zeta)_o\}$$

This gives the stronger ultrametric inequality:

$$d(\xi, \zeta) \leq \max\{d(\xi, \eta), d(\eta, \zeta)\} \quad \square$$

5. CANTOR BOUNDARIES AND COMPACTIFICATION

Theorem 5.1. *Let T be a tree in which every vertex has finite valence at least 3. Then ∂T is the Cantor set.*

Proof. We use the fact that the Cantor set is characterized as the unique nonempty compact metrizable totally disconnected perfect topological space.

A tree without leaves is unbounded, so its boundary is nonempty. We use the metric topology given by Theorem 4.5, so metrizability is automatic.

Compactness. Since ∂T is metrizable, compactness is equivalent to sequential compactness. Let (ξ_i) be a sequence in ∂T , and let α_{ξ_i} be the corresponding rays based at o . Only finitely many edges are incident to o , so we can pass to a subsequence whose rays have the same first edge. The opposite vertex to o also has finite valence, so we can pass to a further subsequence with the same first two edges, and continue. A diagonal subsequence converges to the ray α whose first n edges are the stabilized initial n -edge segments. Thus $\xi_{i_k} \rightarrow [\alpha]$.

No isolated points (perfectness). Given $\xi \in \partial T$ and $n \geq 1$, take a ray α representing ξ and branch away from it after n edges. Since every vertex has valence at least 3, this produces a distinct ray α_n agreeing with α on the first n edges. Hence $[\alpha_n] \rightarrow \xi$.

Totally disconnected. For $\xi \neq \eta$ choose n so that $2^{-n} < d(\xi, \eta)$, and set

$$C_n(\xi) := \{\zeta \in \partial T \mid (\xi \mid \zeta)_o \geq n\}$$

This is the closed ball $\overline{B}_{2^{-n}}(\xi)$, and it is also open. Its complement is open because sufficiently close rays must share the same initial n edges. Thus ξ and η are separated by a clopen set and cannot lie in the same connected component. $\square$

Theorem 5.2 (Compactification). *Let T be a tree of finite valence at least 3. There is a topology on $\bar{T} := T \cup \partial T$ with the following properties:*

- (i) $\bar{T}$ is compact.
- (ii) Its restrictions to T and ∂T agree with their respective metric topologies.
- (iii) T is open and dense in $\bar{T}$.

Proof. For $x \in T$ use the usual metric balls as a neighborhood basis. For $\xi \in \partial T$, choose a ray α_ξ based at o and take as a neighborhood basis the sets:

$$C_n(\xi) := \{x \in T \mid [o, x] \text{ and } \alpha_\xi \text{ agree for length } \geq n\} \\ \cup \{\eta \in \partial T \mid (\xi \mid \eta)_o \geq n\}$$

Each boundary point is represented by a ray, and the vertices of the ray approach the boundary point in this topology, so T is dense. For every point $x \in T$ and every positive r the ball $B_r(x)$ is an open set containing x and contained in T , so T is open.

Compactness is proved by the same diagonal argument as for ∂T . (It turns out that $\bar{T}$ is metrizable, but here it is enough to observe that $\bar{T}$ is second countable, so compact is equivalent to sequentially compact.) Every sequence in ∂T has a convergent subsequence. For a sequence (x_i) in T , either it is bounded, in which case properness of the locally finite tree gives a convergent subsequence, or (after passing to a subsequence) $d(o, x_i) \rightarrow \infty$. In the latter case, apply the diagonal argument to the finite segments $[o, x_i]$ to obtain a limiting ray. $\square$

6. ISOMETRIES OF REGULAR TREES

Let T_r be a regular tree of valence r , where $3 \leq r < \infty$. An *isometry* $\phi: T_r \rightarrow T_r$ preserves distances:

$$d(x, y) = d(\phi(x), \phi(y)) \quad (x, y \in T_r)$$

Notice that vertices have valence at least 3, vertices are distinguishable from interior points of edges. Thus, a map $T_r \rightarrow T_r$ is an isometry if and only if it is an isomorphism of the tree structure, meaning a bijection of the vertices that preserves adjacency.

An isometry is one of the following types:

Elliptic: It fixes a point of T_r . We will show in this case it fixes a vertex or the midpoint of an edge.

Hyperbolic/loxodromic: it has no fixed point in T_r ; it fixes exactly two boundary points ξ^-, ξ^+ , translates along the axis (ξ^-, ξ^+) , and has north-south dynamics on ∂T_r : for every pair of disjoint neighborhoods $U^- \ni \xi^-, U^+ \ni \xi^+$, there exists N such that $\forall n \geq N$:

$$\phi^n(\partial T_r \setminus U^-) \subseteq U^+$$

Also, the axis (ξ^-, ξ^+) is equal to the minimal displacement set of ϕ .

Proof of the classification. If ϕ inverts an edge then it fixes the midpoint of that edge.

Suppose ϕ fixes no vertex and inverts no edge. Suppose u is an interior point of an edge $w - x$. Then $\phi(u) \notin e$, so it is an interior point of some different edge $y - z$. Consider the geodesic from u to $\phi(u)$. It goes through either w or x , and it goes through either y or z . Up to relabelling, suppose it goes through x and y . Then $d(x, y) < d(u, \phi(u))$. Now there are two cases: we have $\phi(\{w, x\}) =$

$\{y, z\}$, so either $\phi(x) = y$ or $\phi(x) = z$. In the former case $d(x, \phi(x)) < d(u, \phi(u))$. In the latter case $d(x, \phi(x)) = d(u, \phi(u))$. Thus, for all u there exists a vertex x such that $d(x, \phi(x)) \leq d(u, \phi(u))$.

Since vertices move by distances that are nonnegative integers, we have shown that the following quantity is a minimum, and it is either 0 and realized at the midpoint of an edge, or it is realized at a vertex:

$$\tau(\phi) := \inf_{u \in T_r} d(u, \phi(u))$$

The elliptic case is $\tau(\phi) = 0$, which is realized at either the midpoint of an edge or at a vertex.

For the loxodromic case, suppose $\tau(\phi) > 0$ and choose $v \in VT_r$ realizing the positive minimum. We claim that the segments $[v, \phi(v)]$ and $[\phi(v), \phi^2(v)]$ do not overlap in a nontrivial segment. Otherwise, let x be the point of $[v, \phi(v)]$ at distance $1/2$ from $\phi(v)$. Then x is also the point of $[\phi(v), \phi^2(v)]$ at distance $1/2$ from $\phi(v)$, and $\phi(x)$ is the point of $[\phi(v), \phi^2(v)]$ at distance $1/2$ from $\phi^2(v)$, so:

$$d(x, \phi(x)) = d(\phi(v), \phi^2(v)) - 1 = d(v, \phi(v)) - 1$$

This contradicts minimality of $d(v, \phi(v))$.

Consequently the segments

$$[\phi^n(v), \phi^{n+1}(v)] \quad (n \in \mathbb{Z})$$

concatenate to form a bi-infinite geodesic. The isometry ϕ translates along this axis, and its two endpoints are

$$\xi^+ := \lim_{n \rightarrow \infty} \phi^n(v) \quad \xi^- := \lim_{n \rightarrow -\infty} \phi^n(v)$$

If η is different from these endpoints, the geodesic ray from σ to $\phi^n(\eta)$ overlaps longer and longer with the ray toward ξ^+ , uniformly for η outside any fixed neighborhood of ξ^- . This is the asserted north-south dynamics.

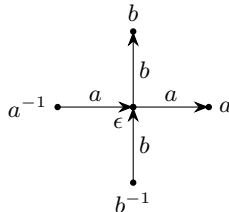
Finally, the axis (ξ^-, ξ^+) is the entire minimal displacement set, as follows. If $u \notin (\xi^-, \xi^+)$ then there is closest point v of (ξ^-, ξ^+) to u . The point v is a vertex, and $[u, v] \cap (\xi^-, \xi^+) = \{v\}$. Apply ϕ : $\phi(v)$ is the closest point of (ξ^-, ξ^+) to $\phi(u)$, $\phi([u, v]) = [\phi(u), \phi(v)]$, and $[\phi(u), \phi(v)] \cap (\xi^-, \xi^+) = \{\phi(v)\}$. Then we claim that v and $\phi(v)$ separate u and $\phi(u)$, so $d(u, \phi(u)) > d(v, \phi(v))$. $\square$

7. THE FREE GROUP AND ITS CAYLEY TREE

The group $F_2 = \langle a, b \mid \rangle$ is the free group on two generators. Its elements are uniquely represented by freely reduced words in the letters a, a^{-1}, b, b^{-1} . The identity is the empty word, denoted ϵ . Multiplication is concatenation followed by free reduction. For example,

$$(aba^{-1})(ab^2) = ab^3$$

The group F_2 acts isometrically on the regular tree T_4 . The action is free and transitive on the vertices, has no edge inversions, and has two orbits of unoriented edges. We establish these facts by constructing a labelling of the tree. Label one vertex by ϵ . It has 4 incident edges. We declare two of them to be outgoing and label them a and b , and declare the other two to be incoming, and label them a^{-1} and b^{-1} . By convention, the label of an edge read against its direction is the inverse of the given label, so from the point of view of ϵ the four outgoing edges are labelled a, b, a^{-1} and b^{-1} . Label the corresponding vertices a, b, a^{-1} and b^{-1} .



Now induct: take a closest vertex to ϵ that is labelled, say by reduced word w , but for which not all of its incident edges are labelled. By construction, exactly one of its incident edges has been directed and labelled, the edge leading to ϵ . Assign directions and labels to the remaining three edges in such a way that vertex w has one incoming and one outgoing edge labelled by each of a and b . Label the adjacent vertices of w as follows. For the outgoing edge labelled a , label the next vertex by the free reduction of wa . For the incoming edge labelled a , label the next vertex by the free reduction of wa^{-1} . Proceed similarly for b . Observe that there is exactly one of these four labels that require free reduction: suppose, for example, that $w = ua$. Then appending a , b , or b^{-1} results in a freely reduced word; only a^{-1} requires a free reduction. Moreover, we arrived at the vertex labelled w from the vertex labelled u by following the outgoing edge at u labelled a , so $wa^{-1} = uaa^{-1} \mapsto u$. Thus, the labels assigned in the induction step are consistent with labels from previous steps.

This process limits to a labelling of the tree in which every vertex is labelled by a freely reduced word of length equal to its distance from ϵ , and every vertex has one incoming and one outgoing edge with each label a and b . In such a labelled tree, an edge path has a label, defined by concatenating the labels of its edges, inverting the label if the edge is traversed against its assigned direction. The label of an embedded edge path is a freely reduced word. Moreover, given a freely reduced word and a vertex, there is a unique edge path starting at that vertex whose label is the given reduced word. It follows that the labelling gives a bijection between vertices of T_4 and elements of F_2 , since elements of F_2 are in bijection with freely reduced words, which are in bijection with embedded edge paths with starting vertex ϵ , thus with vertices of T_4 .

Now we claim that left multiplication of F_2 on the labels of the vertices induces an isometry of T_4 . This is because edges are of the form $x \xrightarrow{a} xa$, so $w \in F_2$ takes the end vertices of that edge to wx and wxa , which differ by right multiplication of a , so there is an edge $wx \xrightarrow{a} wxa$. Thus, F_2 acts on T_4 , freely and transitively on vertices, and preserving the directions and labels of the edges.

Suppose $\epsilon \neq w = uvu^{-1}$ is written as a freely reduced product with v cyclically reduced. Thus there is no cancellation within uvu^{-1} , and the first and last letters of v are not inverses. Multiplication by w has fixed points in ∂T_4 :

$$w^{+\infty} = uv^{+\infty} := \lim_{n \rightarrow \infty} uv^n \quad w^{-\infty} = uv^{-\infty} := \lim_{n \rightarrow \infty} uv^{-n}$$

These are the two ends of the axis of w .

8. CENTRALIZERS IN THE FREE GROUP

Theorem 8.1. *For $w \in F_2 \setminus \{1\}$, the centralizer*

$$Z_{F_2}(w) := \{x \in F_2 \mid xwx^{-1} = w\}$$

is isomorphic to $\mathbb{Z}$.

Proof. Let $x \in Z_{F_2}(w)$. The isometry w translates along the bi-infinite axis:

$$A_w = (w^{-\infty}, w^{+\infty})$$

For any $v \in A_w$, we claim that $xv \in A_w$, because A_w is the entire minimal displacement set of the w -action, and:

$$\tau(w) = d(v, wv) = d(xv, xwv) = d(xv, wxv)$$

Therefore $x(A_w) = A_w$. It cannot interchange the two endpoints of A_w , because then it would fix an interior point of the axis, whereas the F_2 action has no nontrivial elliptic elements. Hence x acts on A_w by translation.

Restriction to the axis gives a homomorphism

$$Z_{F_2}(w) \longrightarrow \text{Trans}(A_w) \cong \mathbb{Z}$$

Its kernel is trivial because the action of F_2 on vertices of T_4 is free. The image is a subgroup of $\mathbb{Z}$ and is nontrivial because it contains w . Hence the centralizer is isomorphic to $\mathbb{Z}$. $\square$

9. PING-PONG ON THE BOUNDARY

Theorem 9.1 (Ping-pong). *Let $f, g \in F_2 \setminus \{1\}$ satisfy*

$$|\{f^{+\infty}, f^{-\infty}, g^{+\infty}, g^{-\infty}\}| = 4$$

Then, for some $n \geq 1$, the subgroup generated by f^n and g^n is isomorphic to F_2 .

Proof. Choose pairwise disjoint neighborhoods

$$A^+ \ni f^{+\infty}, \quad A^- \ni f^{-\infty}, \quad B^+ \ni g^{+\infty}, \quad B^- \ni g^{-\infty}$$

which do not together cover all of ∂T_4 . By north-south dynamics, choose n large enough that

$$\begin{aligned} f^n(\partial T_4 \setminus A^-) &\subseteq A^+ \\ f^{-n}(\partial T_4 \setminus A^+) &\subseteq A^- \\ g^n(\partial T_4 \setminus B^-) &\subseteq B^+ \\ g^{-n}(\partial T_4 \setminus B^+) &\subseteq B^- \end{aligned}$$

Set $\alpha = f^n$ and $\beta = g^n$. We wish to prove that every nonempty freely reduced word w in $\alpha^{\pm 1}, \beta^{\pm 1}$ is nontrivial. Choose $\xi \in \partial T_4$ outside all four sets A^+, A^-, B^+, B^- . It suffices to show that $w\xi \neq \xi$.

The point ξ is the “ping-pong ball”, and the game is that the letters of w hit the ball into their designated regions: If, for example, the last letter of w is α , so $w = w'\alpha$, then the first player is α , and, since $\xi \notin A^-$, α hits ξ into A^+ , ie, $\alpha\xi \in A^+$. The next player is the last letter of w' . Since w is freely reduced, the next player is not α^{-1} , it is one of α, β , or β^{-1} . Since the ball currently lies in A^+ , it is not in A^-, B^+ , or B^- , so player 2 hits the ball into A^+, B^+ , or B^- , respectively. Continue, inducting on the length of w , to see that from the first hit onwards, the ball is always in one of the sets A^+, A^-, B^+ , or B^- , so it never returns to the starting point ξ . Thus, $w\xi \neq \xi$, so $w \neq 1$. $\square$

10. EXERCISES

1. Prove Lemma 2.1.
2. Prove Lemma 3.5.
3. Prove Lemma 4.4.
4. **Gromov products and the four-point condition.**

Let T be a tree and $o \in T$.

- (a) Show that for every $x, y, z \in T$:

$$(x \mid z)_o \geq \min\{(x \mid y)_o, (y \mid z)_o\}$$

- (b) Show that for any four points $x, y, z, w \in T$, the two largest of the following quantities are equal:

$$d(x, y) + d(z, w)$$

$$d(x, z) + d(y, w)$$

$$d(x, w) + d(y, z)$$

- (c) Interpret this exercise as a statement about geodesic quadrilaterals in a tree.

5. **Nearest-point projections.**

Let C be a nonempty connected subgraph of a tree T .

- (a) Show C is a tree.
- (b) Show that every $x \in T$ has a unique closest point $\pi_C(x) \in C$.
- (c) Show that for every $y \in C$, the geodesic $[x, y]$ passes through $\pi_C(x)$.

- (d) Show that $\pi_C: T \rightarrow C$ is 1-Lipschitz. A map ϕ is L -Lipschitz if $d(\phi(x), \phi(y)) \leq Ld(x, y)$ for all x and y .

6. The Morse lemma for trees.

A (not necessarily continuous!) map $\gamma: I \rightarrow X$ is a (λ, ε) -quasigeodesic if $\lambda \geq 1$, $\varepsilon \geq 0$, and for all s, t in the interval I :

$$\frac{1}{\lambda}|s - t| - \varepsilon \leq d(\gamma(s), \gamma(t)) \leq \lambda|s - t| + \varepsilon$$

Suppose γ is a continuous (λ, ε) -quasigeodesic from x to y in a tree T .

- (a) Use Lemma 3.6 to show that $[x, y]$ is contained in the image of γ .
 (b) For $z \in \gamma([a, b]) \setminus [x, y]$, let p be its closest point on $[x, y]$. Show that there exist $s < t < u$ such that $\gamma(s) = \gamma(u) = p$ and $\gamma(t) = z$.
 (c) Deduce that:

$$d(z, [x, y]) \leq \frac{\lambda^2 \varepsilon}{2} + \varepsilon$$

- (d) Show that every $(\lambda, 0)$ -quasigeodesic in a tree has the same image as the geodesic between its endpoints.

7. Changing the basepoint.

Let T be a tree and let $o, o' \in VT$.

- (a) Show that for $\xi, \eta \in \partial T$:

$$|(\xi | \eta)_o - (\xi | \eta)_{o'}| \leq d(o, o')$$

- (b) Deduce that the two metrics given by Theorem 4.5 satisfy:

$$2^{-d(o, o')} d_o(\xi, \eta) \leq d_{o'}(\xi, \eta) \leq 2^{d(o, o')} d_o(\xi, \eta)$$

- (c) Conclude that the topology of ∂T is independent of the choice of basepoint.

8. Translation lengths.

Let g be a hyperbolic isometry of T_r , with axis A_g .

- (a) Prove that for every $n \in \mathbb{Z}$:

$$\tau(g^n) = |n| \tau(g)$$

- (b) Prove that for every $x \in T_r$ and every positive integer n :

$$d(x, g^n x) = n \tau(g) + 2d(x, A_g)$$

- (c) Deduce that:

$$\lim_{n \rightarrow \infty} \frac{d(x, g^n x)}{n} = \tau(g)$$

9. North-south dynamics.

Give a detailed proof of the north-south dynamics assertion from the classification of isometries of T_r .

10. Common boundary fixed points.

Let $f, g \in F_2 \setminus \{1\}$.

- (a) Suppose that f and g have a common fixed point in ∂T_4 . Show that their axes have a common infinite ray.
 (b) Show that the commutator $[f, g]$ fixes a vertex of T_4 .
 (c) Deduce that f and g commute.
 (d) Conclude that f and g belong to a common cyclic subgroup of F_2 , and that they have the same two fixed points in ∂T_4 .
 (e) Deduce from Theorem 9.1 that if f and g do not belong to a common cyclic subgroup, then for sufficiently large n :

$$\langle f^n, g^n \rangle \cong F_2$$